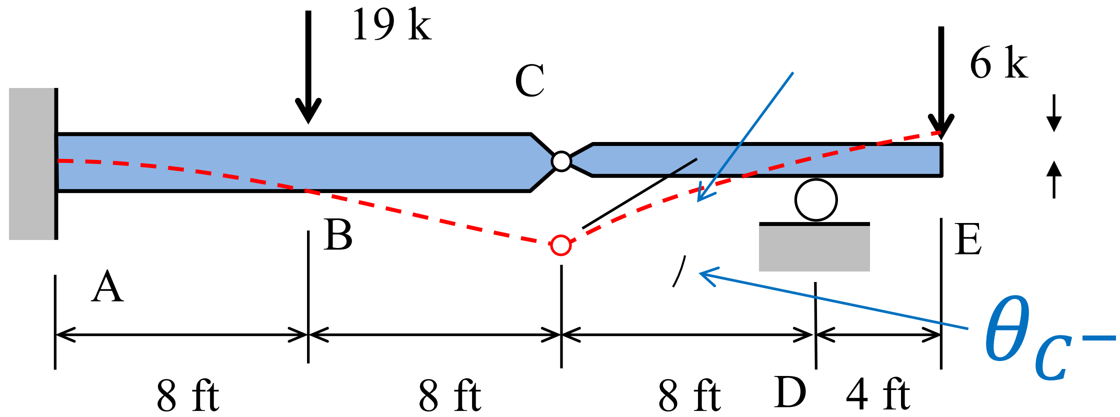
Method of Virtual Work
Frame Deflection Example

Steven Vukazich
San Jose State University

Frame Deflection Example

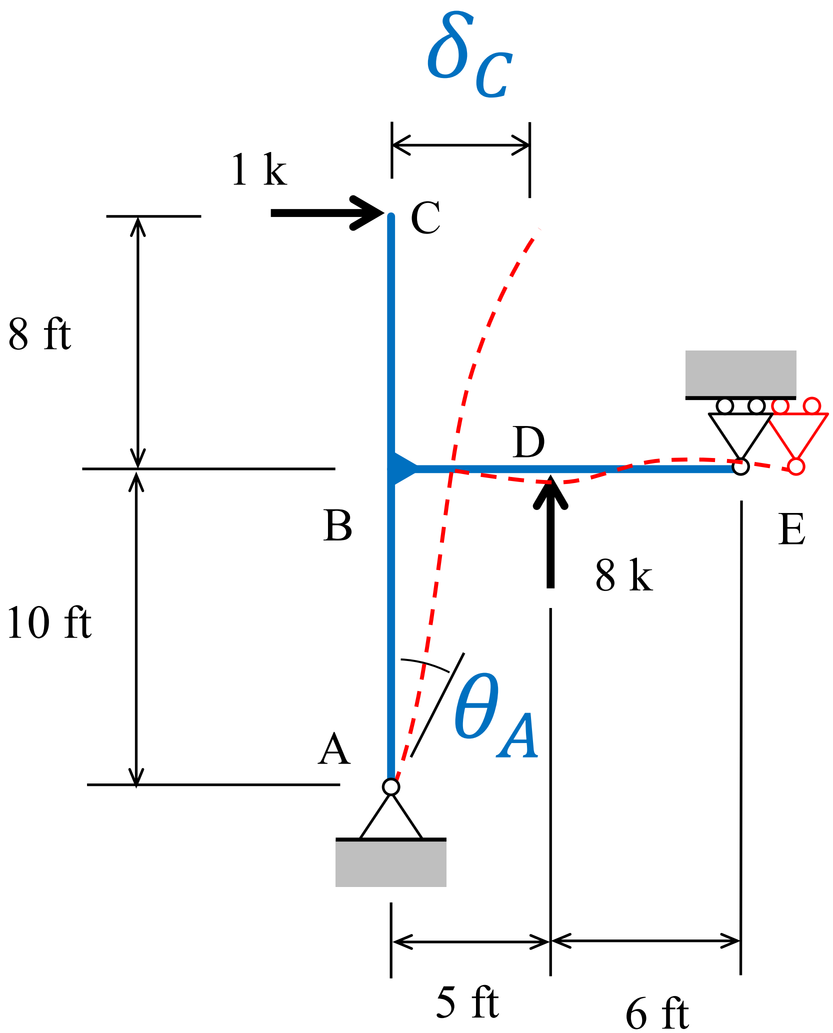


The statically determinate frame from our previous internal force diagram example is made up of columns that are W8x48 (I = 184 in^4) members and a beam that is a W10x22 (I = 118 in^4). The modulus of elasticity of structural steel is 29,000 ksi, which yields the following bending stiffnesses:

- EI_{ABC} = 5,336,000 k-in^2
- EI_{BDE} = 3,422,000 k-in^2

For the loads shown, find the following:

Frame Deflection Example



The statically determinate frame from our previous internal force diagram example is made up of columns that are W8x48 ($I = 184\ in^4$) members and a beam that is a W10x22 ($I = 118\ in^4$). The modulus of elasticity of structural steel is 29,000 ksi, which yields the following bending stiffenesses:

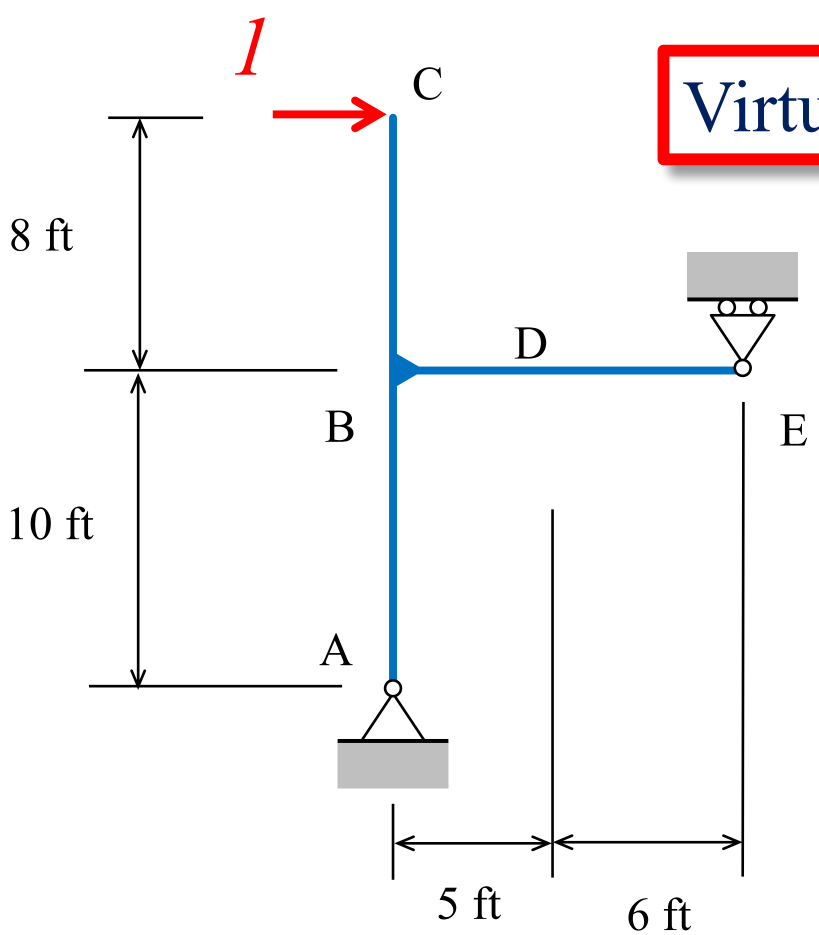
- $EI_{ABC} = 5{,}336{,}000$ k-in^2
- $EI_{BDE} = 3{,}422{,}000$ k-in^2

For the loads shown, find the following:

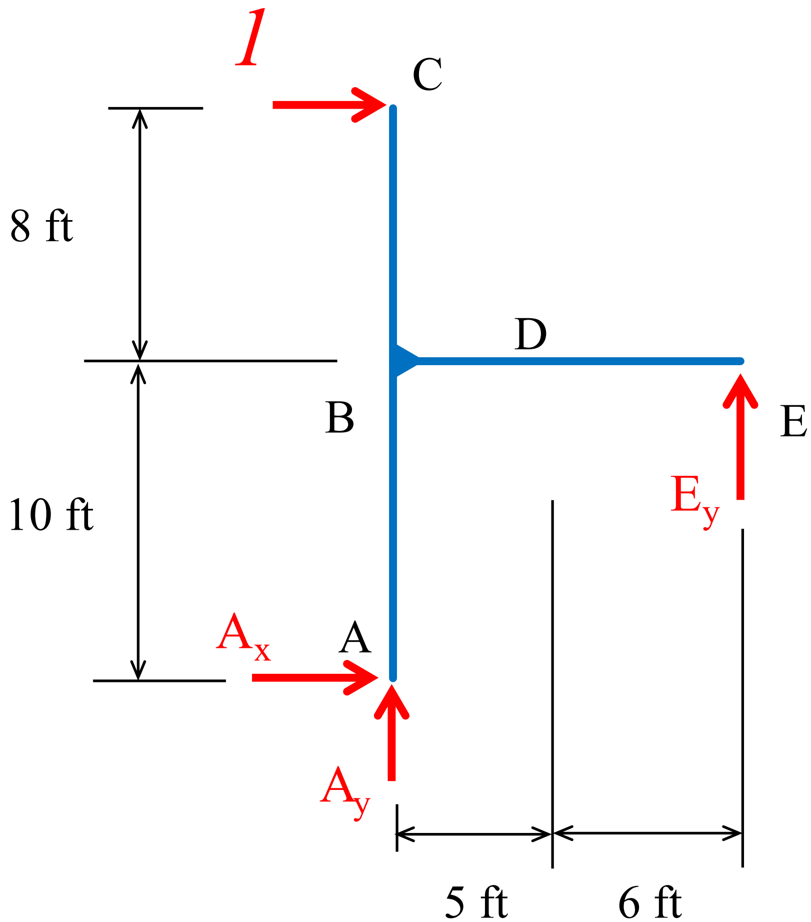
1. The horizontal deflection at point C;
2. The slope at the pin support at A.

Find the Horizontal Deflection at Point C

Virtual system to measure δ_C



Find Support Reactions for the Virtual System

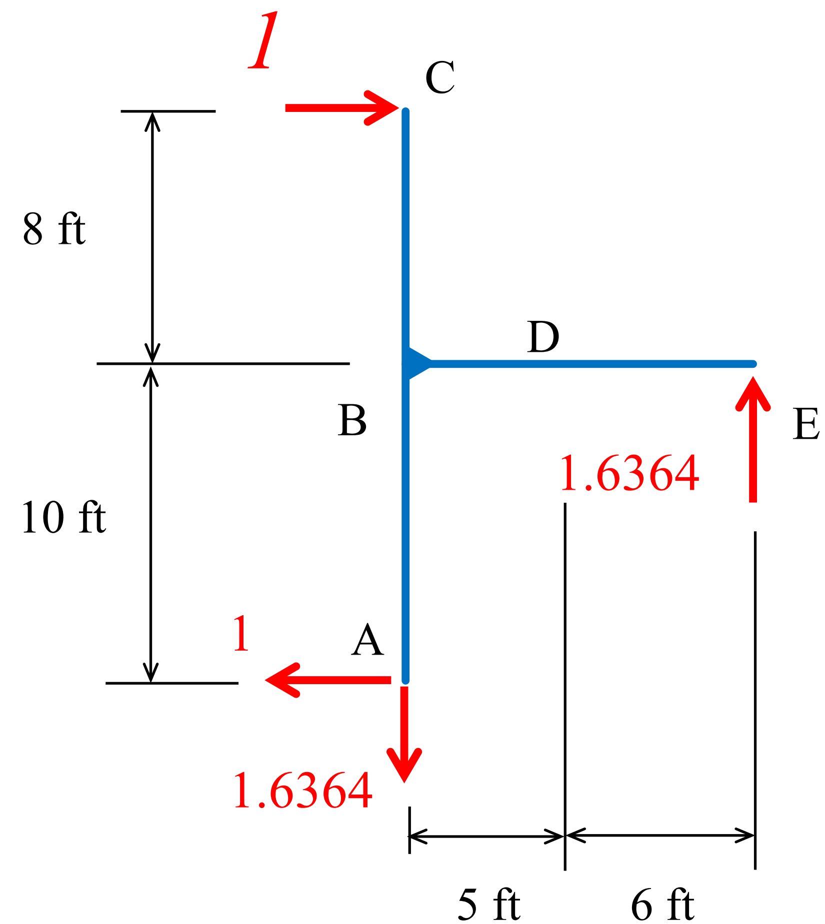


$$\circlearrowleft + \sum M_A = 0 \Rightarrow E_y = 1.6364$$

$$\overset{+}{\rightarrow} \sum F_x = 0 \Rightarrow A_x = -1$$

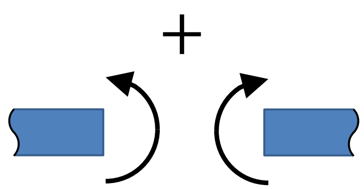
$$+\uparrow \sum F_y = 0 \Rightarrow A_y = -1.6364$$

Support Reactions for the Virtual System

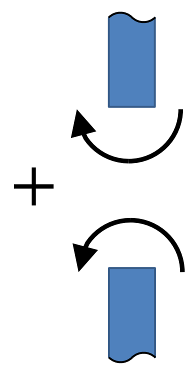


Choose Sign Convention for Internal moment

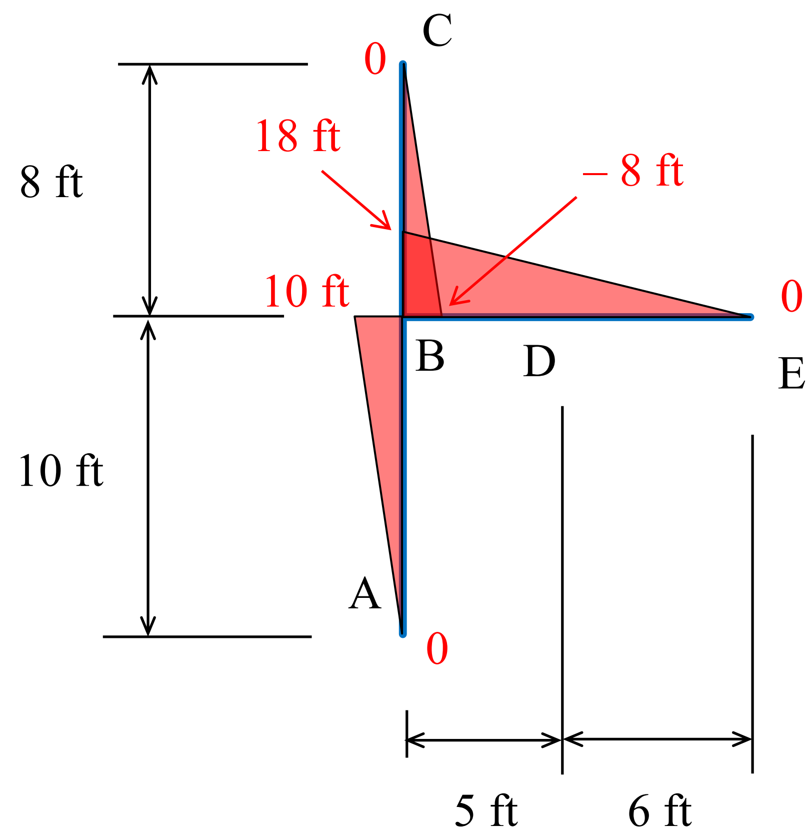
For horizontal member BDE



For vertical member ABC

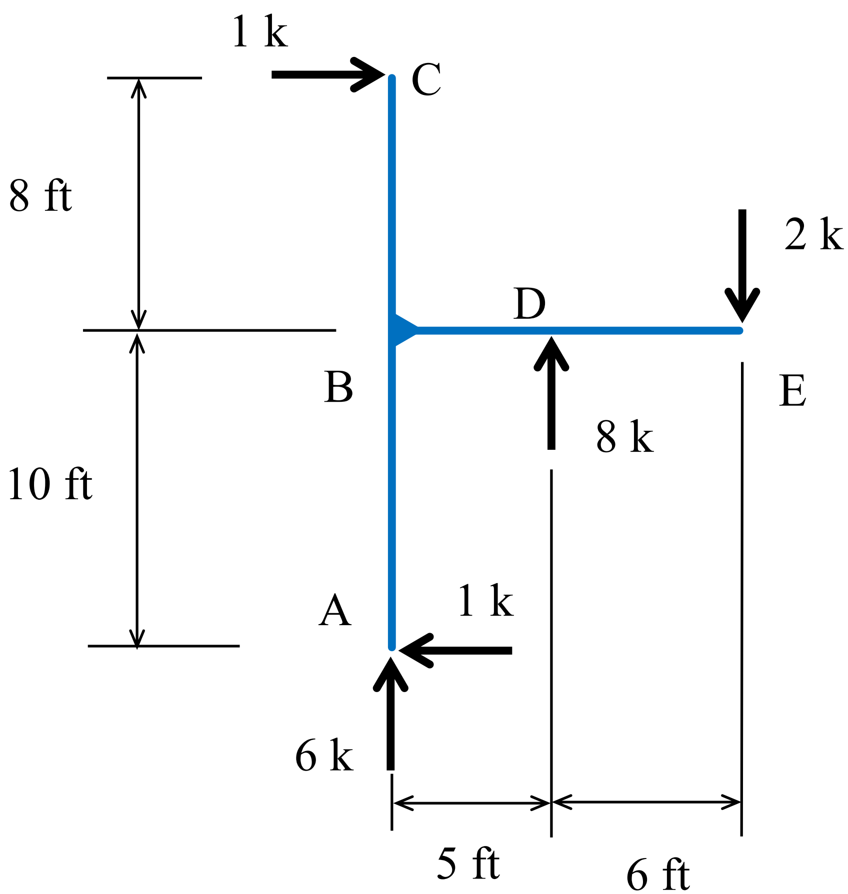


Moment Diagram for the Virtual System



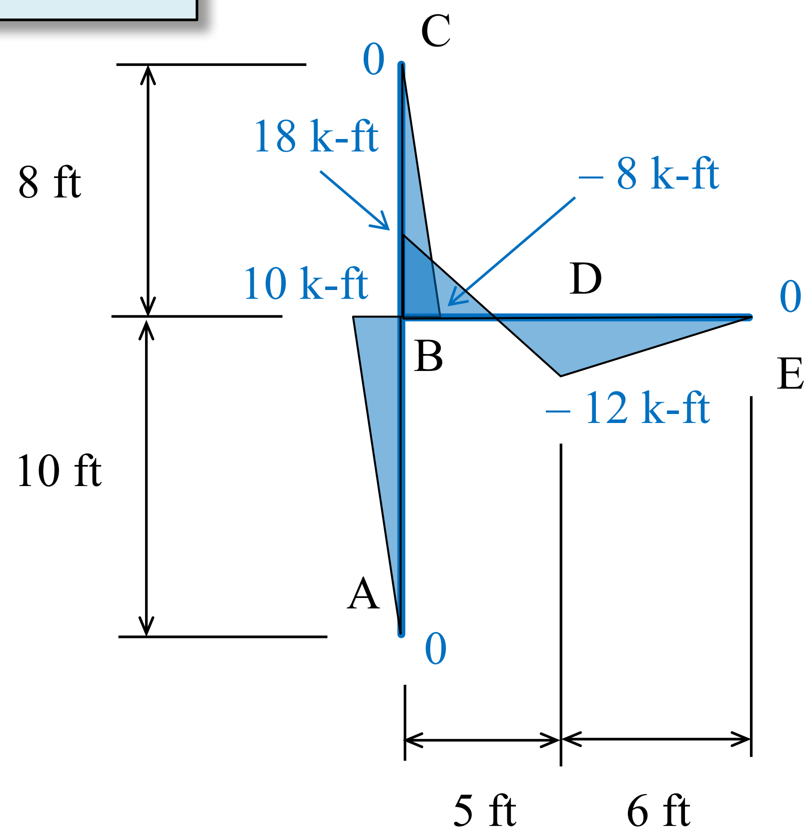
Support Reactions for the Real System

Results from our previous module:

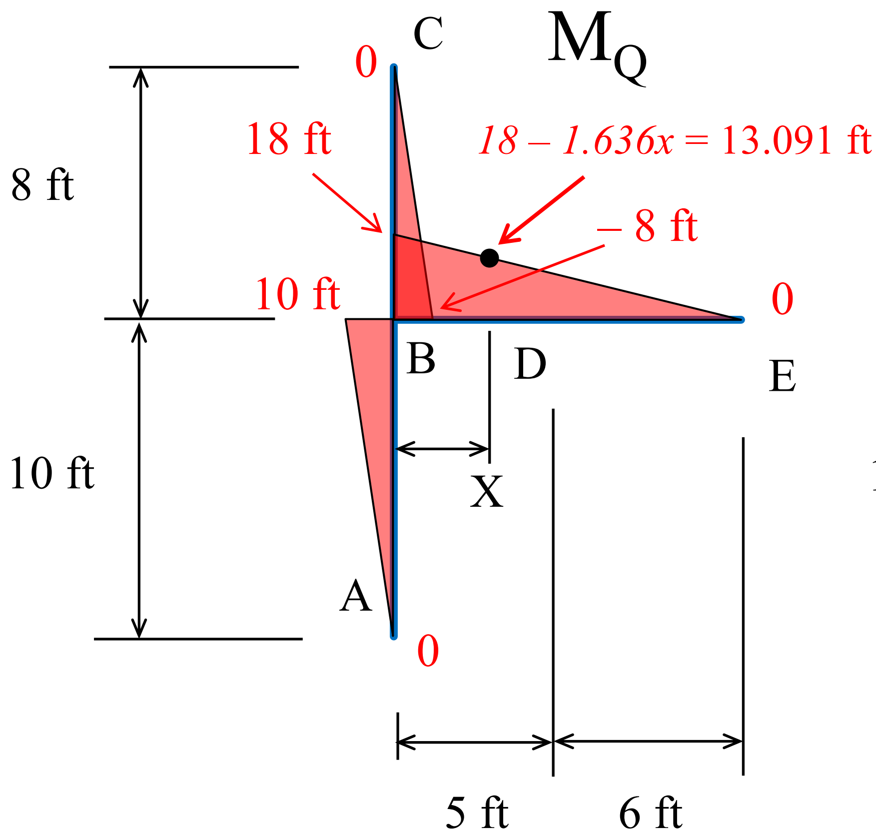


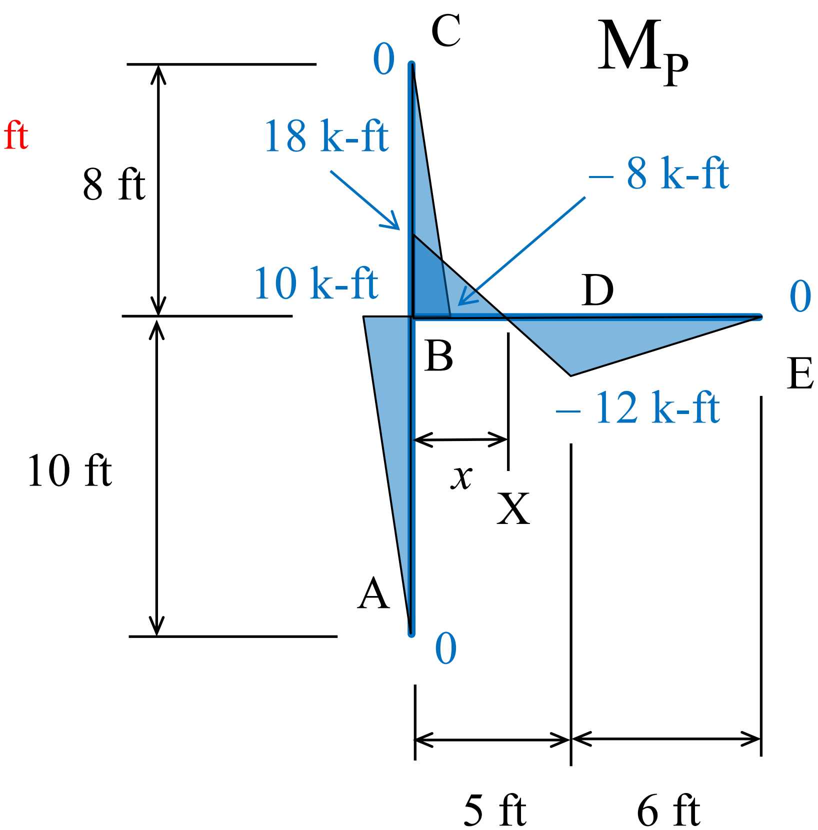
Moment Diagram for the Real System

Results from our previous module:



Evaluate the Virtual Work Product Integrals





$EI_{ABC} = 5{,}336{,}000$ k-in^2

$EI_{BDE} = 3{,}422{,}000$ k-in^2

$$1 \cdot \delta_C = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

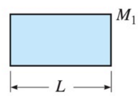
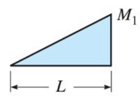
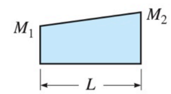
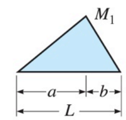
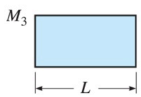
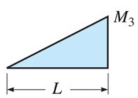
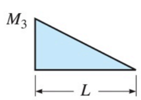
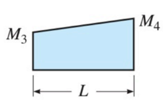
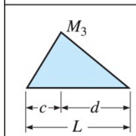
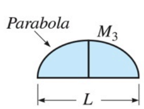
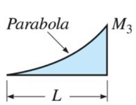
Use Table to evaluate product integrals

$$\frac{18}{x} = \frac{12}{5 - x} \qquad 90 = 30x$$

$$x = 3 \text{ ft}$$

Table to Evaluate Virtual Work Product Integrals

Appendix Table.2

M_P \ M_Q				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L + \frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b) + \frac{1}{6}M_1M_4(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d) + \frac{1}{6}M_2M_3(L+c)$	for $c \le a$: $\left(\frac{1}{3} - \frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

AB and BC

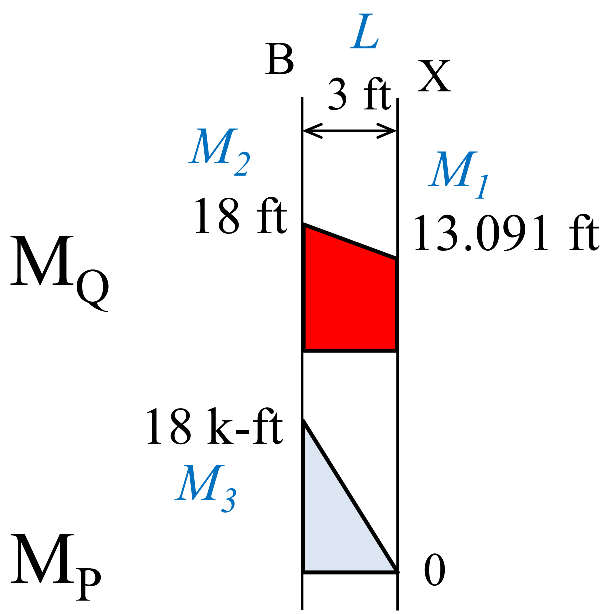
BX

XDE

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

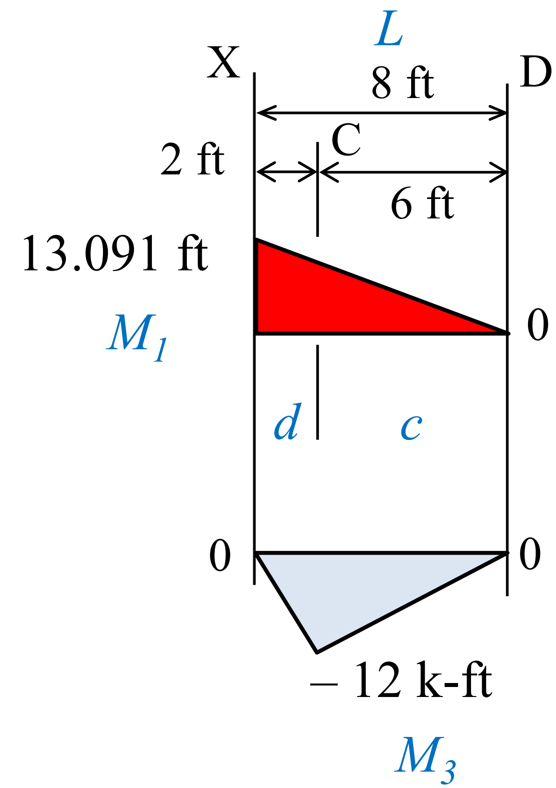
Evaluate Product Integrals



$$\frac{1}{6}(M_1 + 2M_2)M_3L$$

$$\frac{1}{6}\big(13.091 + 2(18)\big)(18)(3)$$

441.819 k-ft^3

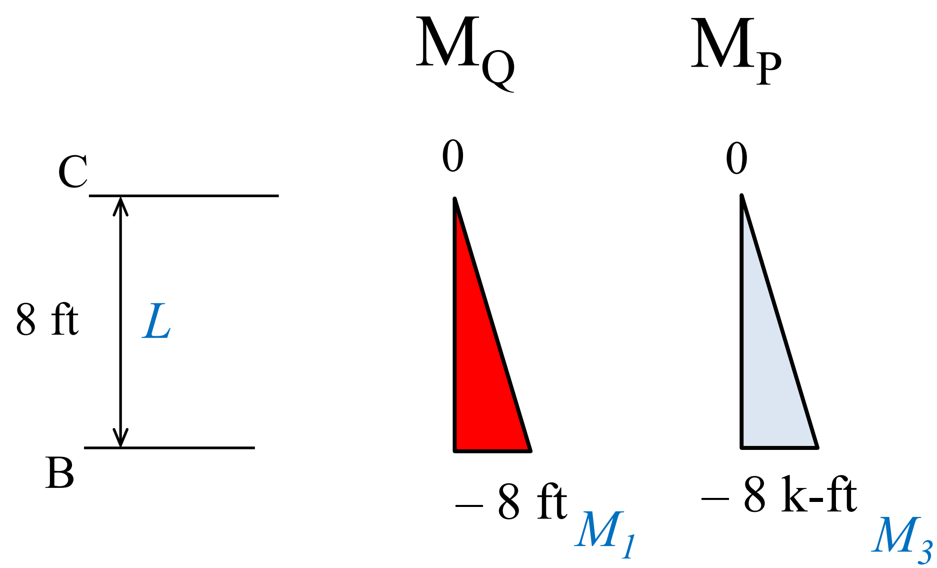


$$\frac{1}{6}M_1M_3(L + c)$$

$$\frac{1}{6}(13.091)(-12)(8 + 6)$$

– 366.548 k-ft^3

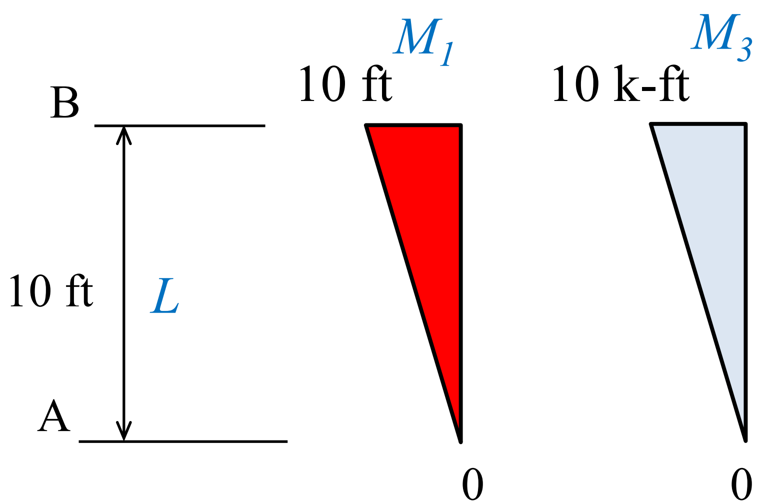
Evaluate Product Integrals



$$\frac{1}{3}M_1M_3L$$

$$\frac{1}{3}(-8)(-8)(8)$$

170.667 k-ft^3



$$\frac{1}{3}M_1M_3L$$

$$\frac{1}{3}(10)(10)(10)$$

333.333 k-ft^3

Evaluate Product Integrals

$$1 \cdot \delta_C = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

Segment BX

441.819 k-ft^3

Segment XDE

– 366.548 k-ft^3

$$\int_0^{L_{BDE}} M_Q\, M_P\, dx = \left(441.819\ \text{k–ft}^3 - 366.548\ \text{k–ft}^3\right)\left(\frac{12^3\ \text{in}^3}{\text{ft}^3}\right)$$
$$= 130{,}068.3\ \text{k–in}^3$$

Segment AB

333.333 k-ft^3

Segment BC

170.667 k-ft^3

$$\int_0^{L_{ABC}} M_Q\, M_P\, dx = \left(333.333 + 170.667\ \text{k–ft}^3\right)\left(\frac{12^3\ \text{in}^3}{\text{ft}^3}\right)$$
$$= 870{,}912\ \text{k–in}^3$$

Evaluate Product Integrals

$$\int_0^{L_{ABC}} M_Q\, M_P dx = (333.333 + 170.667 \text{ k–ft}^3)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = 870{,}912 \text{ k–in}^3$$

$$\int_0^{L_{BDE}} M_Q\, M_P dx = \left(441.819 \text{ k–ft}^3 - 366.548 \text{ k–ft}^3\right)\left(\frac{12^3 \text{ in}^3}{\text{ft}^3}\right) = 130{,}068.3 \text{ k–in}^3$$

$$1 \cdot \delta_C = \frac{1}{EI_{ABC}} \int_0^{L_{ABC}} M_Q\, M_P dx + \frac{1}{EI_{CDE}} \int_0^{L_{BDE}} M_Q\, M_P dx$$

$$\delta_C = \frac{870{,}912 \text{ k–in}^3}{5{,}336{,}000 \text{ k–in}^2} + \frac{130{,}068.3 \text{ k–in}^3}{3{,}422{,}000 \text{ k–in}^2}$$

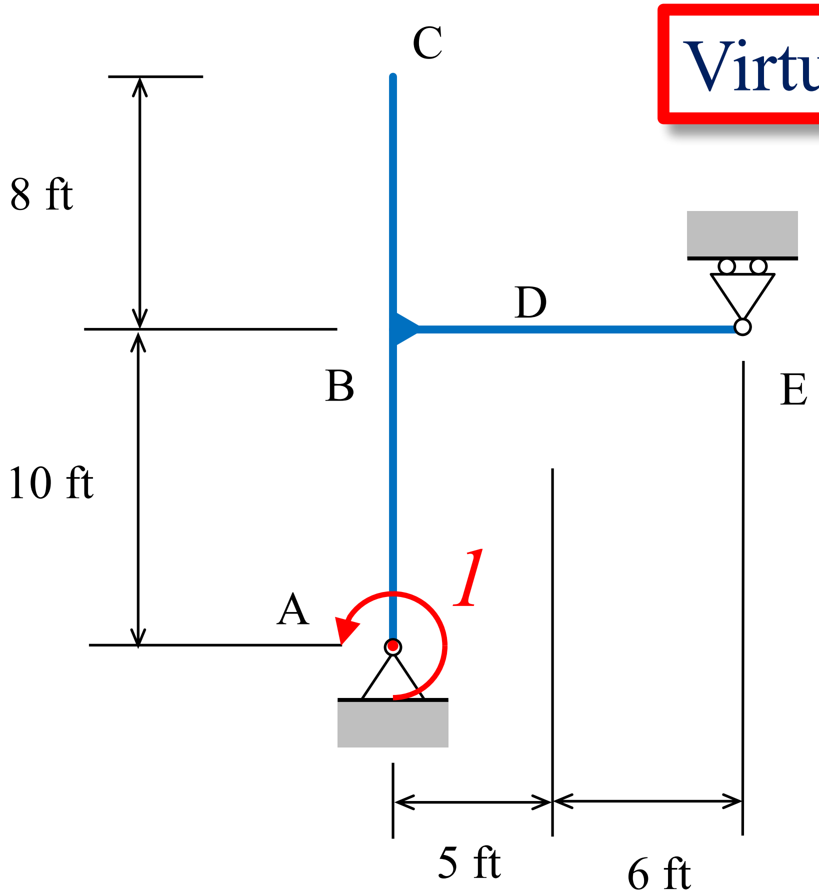
$$\delta_C = 0.163 \text{ in} + 0.0380 \text{ in} = 0.201 \text{ in}$$

Positive result, so deflection is in the same direction as the virtual unit load

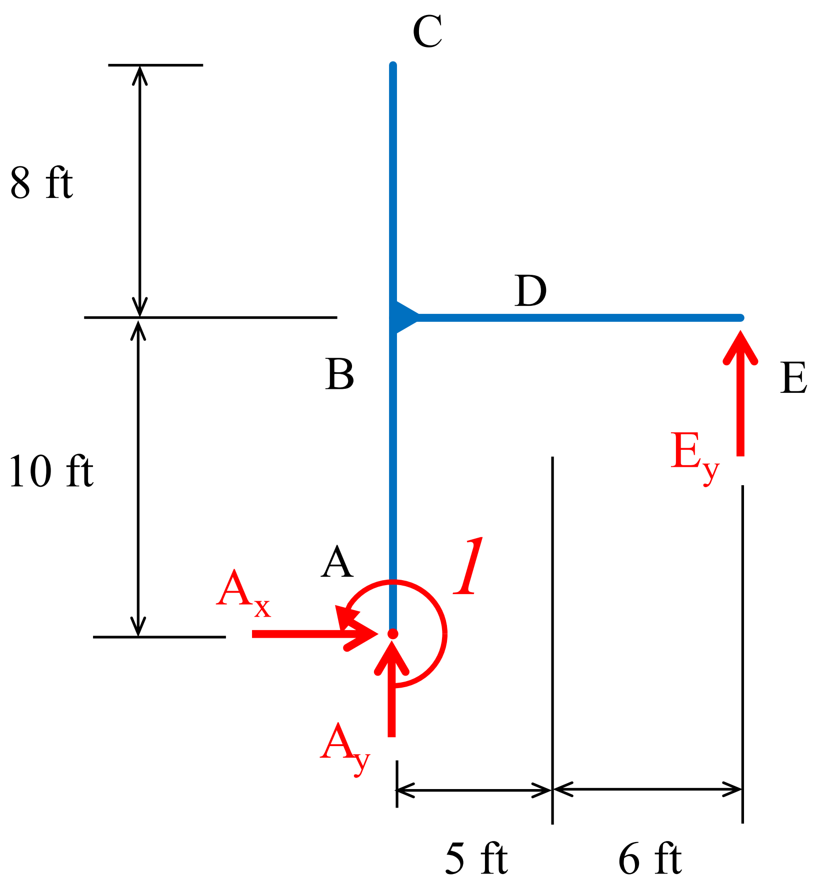
$$\delta_C = 0.201 \text{ inches to the right}$$

Find the the Rotation at Point A

Virtual system to measure θ_A



Find Support Reactions for the Virtual System

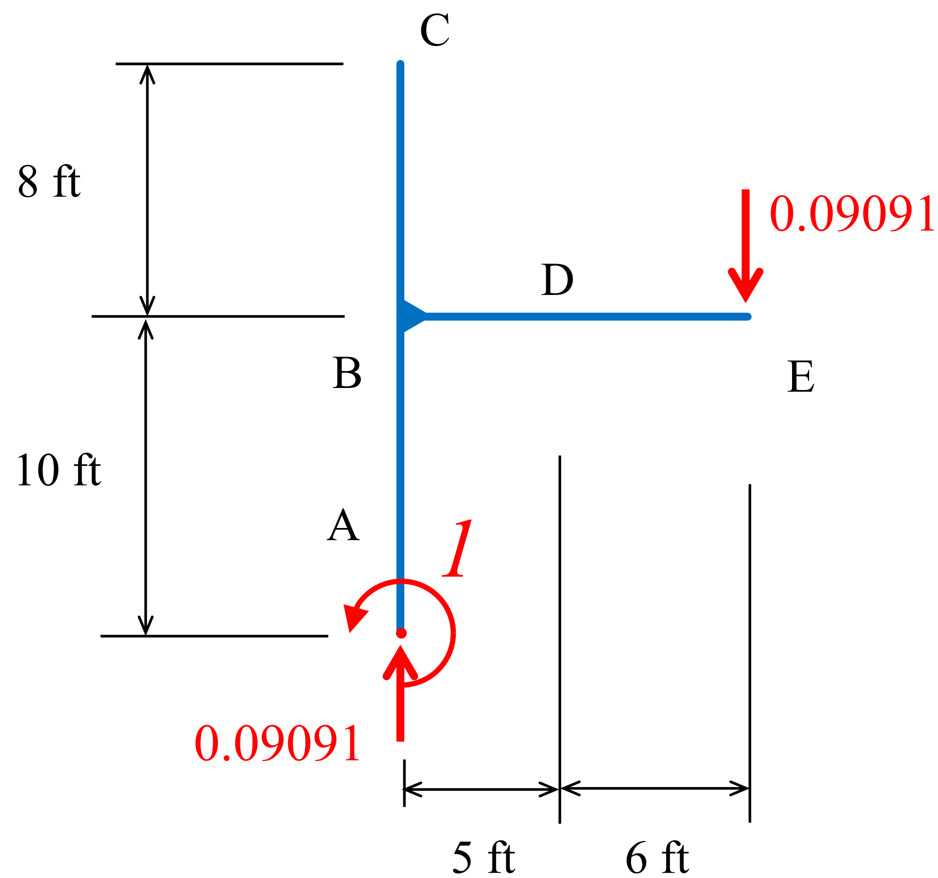


$$\circlearrowleft + \sum M_A = 0 \Rightarrow E_y = -0.09091$$

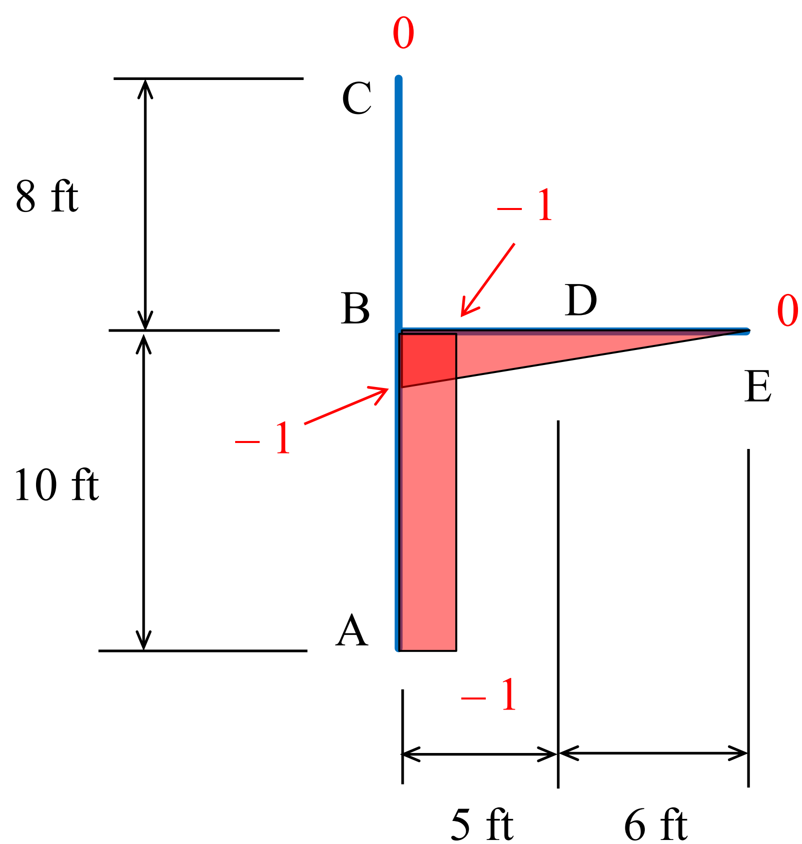
$$\xrightarrow{+} \sum F_x = 0 \Rightarrow A_x = 0$$

$$+\uparrow \sum F_y = 0 \Rightarrow A_y = 0.09091$$

Support Reactions for the Virtual System

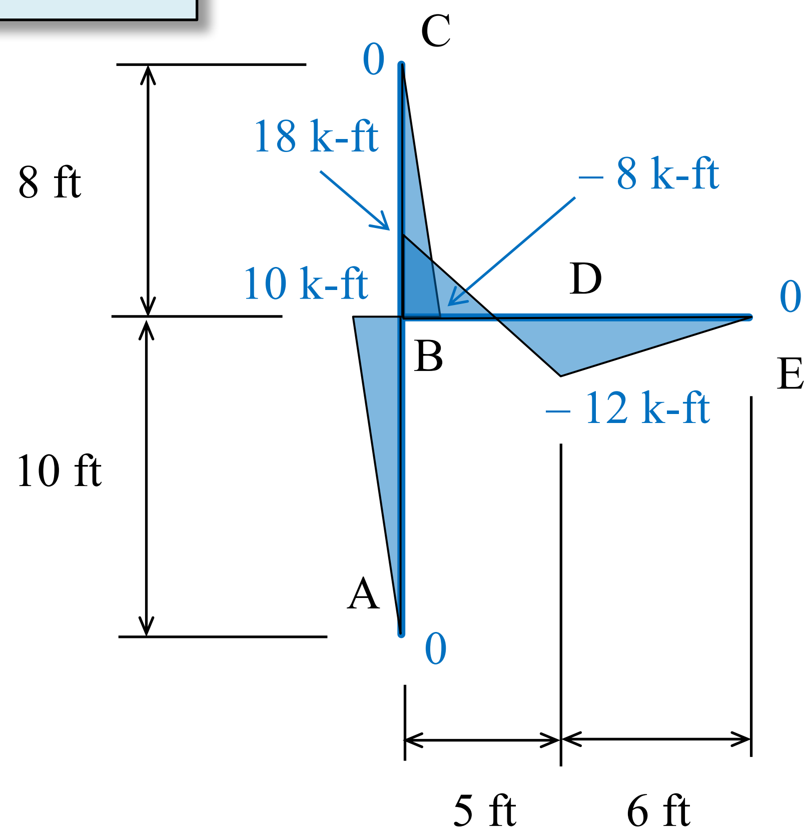


Moment Diagram for the Virtual System

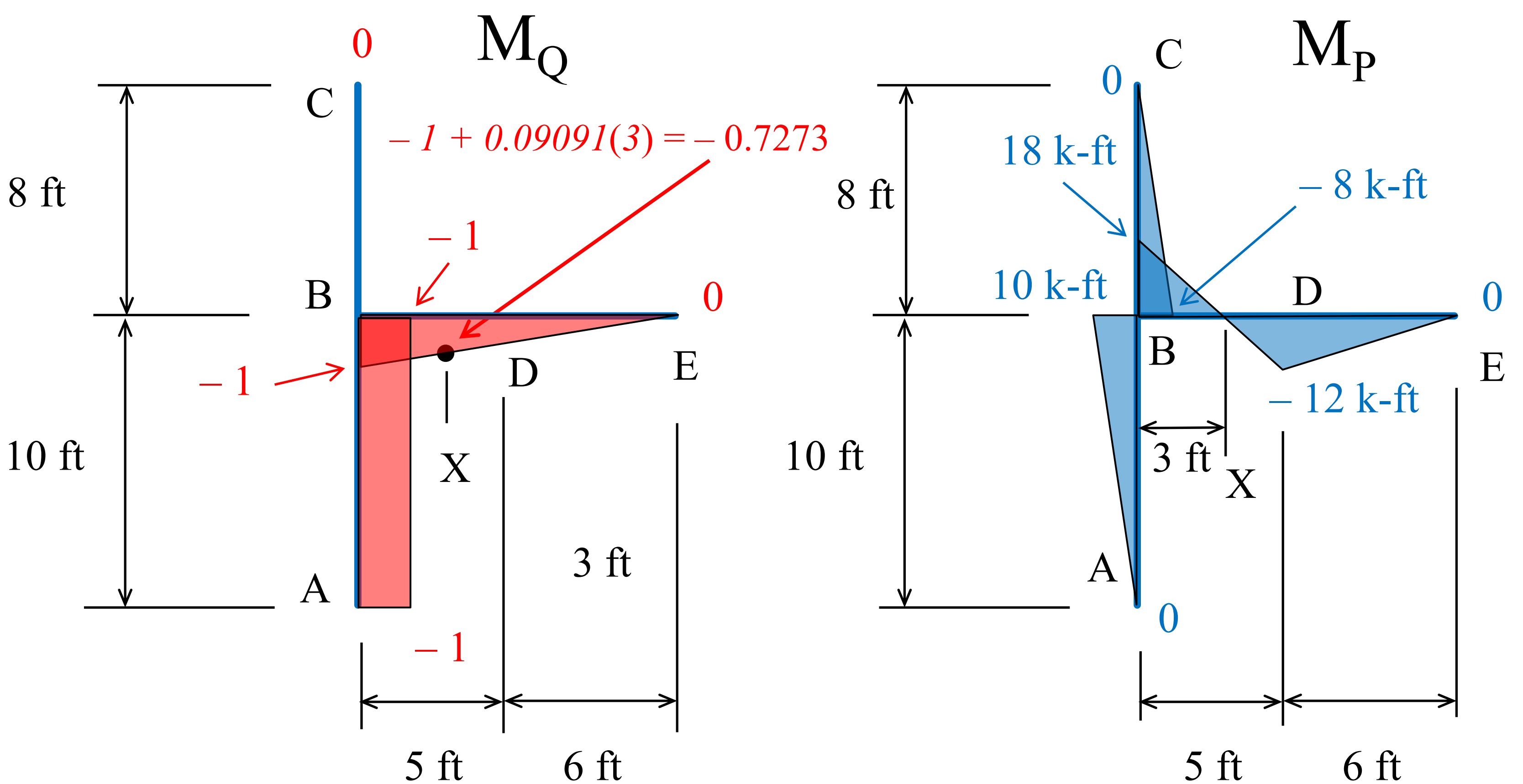


Moment Diagram for the Real System

Results from our previous module:



Evaluate the Virtual Work Product Integrals



$EI_{ABC} = 5{,}336{,}000$ k-in^2

$EI_{BDE} = 3{,}422{,}000$ k-in^2

$$1 \cdot \delta_C = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

Use Table to evaluate product integrals

Table to Evaluate Virtual Work Product Integrals

Appendix Table.2

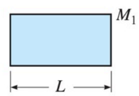
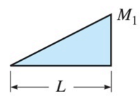
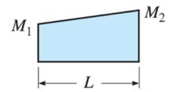
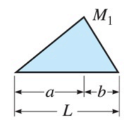
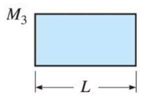
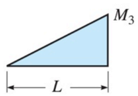
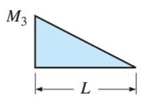
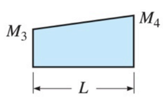
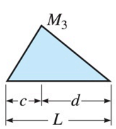
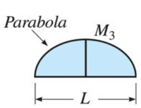
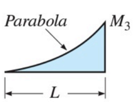
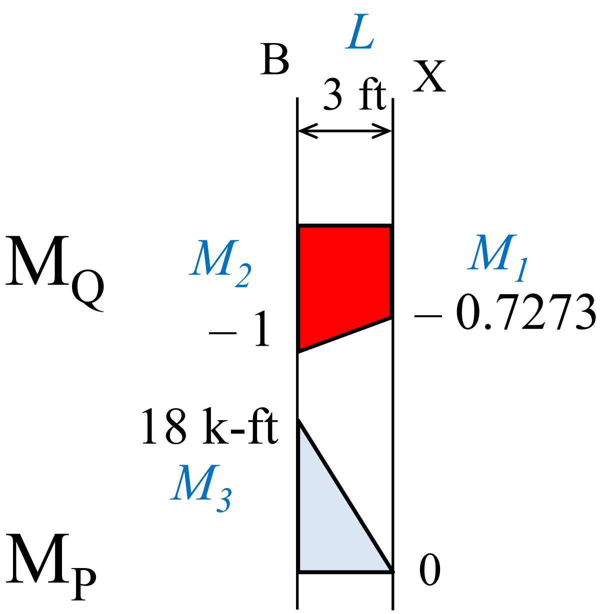
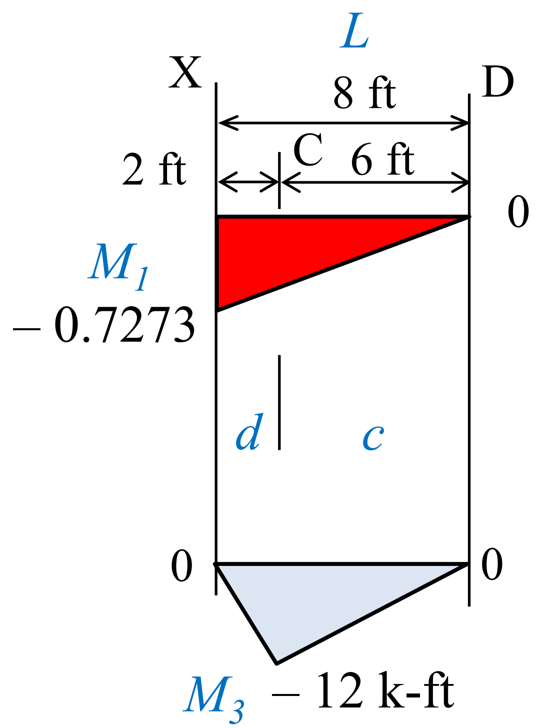
M_Q / M_P				
	M_1M_3L	$\frac{1}{2}M_1M_3L$	$\frac{1}{2}(M_1+M_2)M_3L$	$\frac{1}{2}M_1M_3L$
	AB: $\frac{1}{2}M_1M_3L$	$\frac{1}{3}M_1M_3L$	BX: $\frac{1}{6}(M_1+2M_2)M_3L$	$\frac{1}{6}M_1M_3(L+a)$
	$\frac{1}{2}M_1M_3L$	$\frac{1}{6}M_1M_3L$	$\frac{1}{6}(2M_1+M_2)M_3L$	$\frac{1}{6}M_1M_3(L+b)$
	$\frac{1}{2}M_1(M_3+M_4)L$	$\frac{1}{6}M_1(M_3+2M_4)L$	$\frac{1}{6}M_1(2M_3+M_4)L$ $+\frac{1}{6}M_2(M_3+2M_4)L$	$\frac{1}{6}M_1M_3(L+b)$ $+\frac{1}{6}M_1M_4(L+a)$
	$\frac{1}{2}M_1M_3L$	XDE: $\frac{1}{6}M_1M_3(L+c)$	$\frac{1}{6}M_1M_3(L+d)$ $+\frac{1}{6}M_2M_3(L+c)$	for $c \le a$: $\left(\frac{1}{3}-\frac{(a-c)^2}{6ad}\right)M_1M_3L$
	$\frac{2}{3}M_1M_3L$	$\frac{1}{3}M_1M_3L$	$\frac{1}{3}(M_1+M_2)M_3L$	$\frac{1}{3}M_1M_3\left(L+\frac{ab}{L}\right)$
	$\frac{1}{3}M_1M_3L$	$\frac{1}{4}M_1M_3L$	$\frac{1}{12}(M_1+3M_2)M_3L$	$\frac{1}{12}M_1M_3\left(3a+\frac{a^2}{L}\right)$

Table is as useful tool to evaluate product integrals of the form:

$$\int_0^L M_Q\, M_P\, dx$$

Evaluate Product Integrals





$$\frac{1}{6}(M_1 + 2M_2)M_3L$$

$$\frac{1}{6}\big(-0.7273 + 2(-1)\big)(18)(3)$$

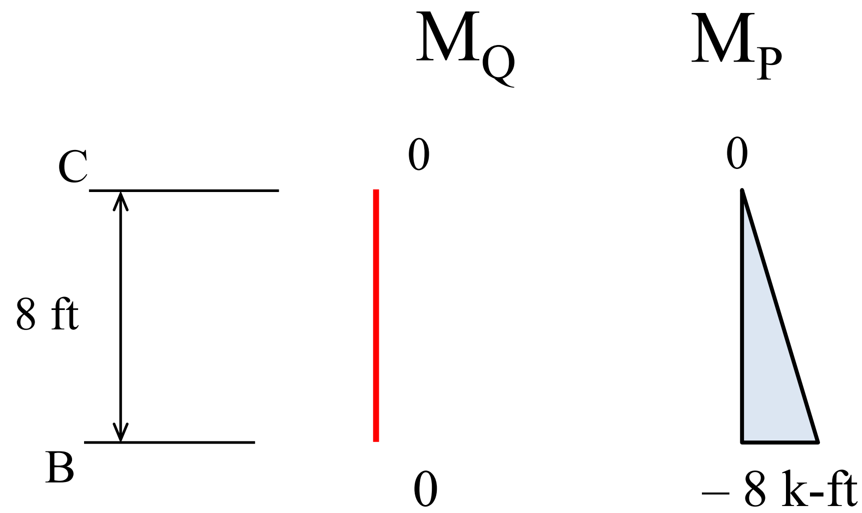
– 24.54545 k-ft²

$$\frac{1}{6}M_1M_3(L + c)$$

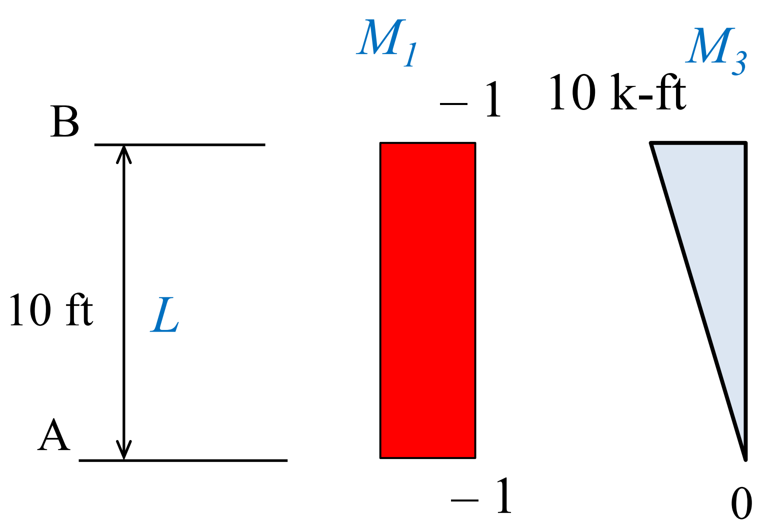
$$\frac{1}{6}(-0.7273)(-12)(8 + 6)$$

20.36364 k-ft²

Evaluate Product Integrals



0



$$\frac{1}{2} M_1 M_3 L$$

$$\frac{1}{2}(-1)(10)(10)$$

– 50.0 k-ft^2

Evaluate Product Integrals

$$1 \cdot \delta_C = \frac{1}{EI}\int_0^L M_Q\, M_P\, dx$$

Segment BX

-24.54545 k-ft^2

Segment XDE

20.36364 k-ft^2

$$\int_0^{L_{BDE}} M_Q\, M_P\, dx = \left(-24.54545 + 20.36364 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right)$$
$$= -602.1818 \text{ k–in}^2$$

Segment AB

-50.0 k-ft^2

Segment BC

0

$$\int_0^{L_{ABC}} M_Q\, M_P\, dx = \left(-50 + 0 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right)$$
$$= -7{,}200 \text{ k–in}^2$$

Evaluate Product Integrals

$$\int_0^{L_{BDE}} M_Q\, M_P dx = \left(-24.54545 + 20.36364 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = -602.1818 \text{ k–in}^2$$

$$\int_0^{L_{ABC}} M_Q\, M_P dx = \left(-50 + 0 \text{ k–ft}^2\right)\left(\frac{12^2 \text{ in}^2}{\text{ft}^2}\right) = -7{,}200 \text{ k–in}^2$$

$$1 \cdot \theta_A = \frac{1}{EI_{ABC}} \int_0^{L_{ABC}} M_Q\, M_P dx + \frac{1}{EI_{BDE}} \int_0^{L_{BDE}} M_Q\, M_P dx$$

$$\theta_A = \frac{-7{,}200 \text{ k–in}^2}{5{,}336{,}000 \text{ k–in}^2} + \frac{-602.1818 \text{ k–in}^2}{3{,}422{,}000 \text{ k–in}^2}$$

$$\theta_A = -0.001349 \text{ rad} - 0.0001760 \text{ rad} = -0.00152 \text{ rad}$$

$$\theta_A = 0.00152 \text{ radians clockwise}$$

Negative result, so rotation is in the opposite direction of the virtual unit moment

Frame Deflection Example Results