Lecture 9-3/8/10-14
Spatial Description and Transformation

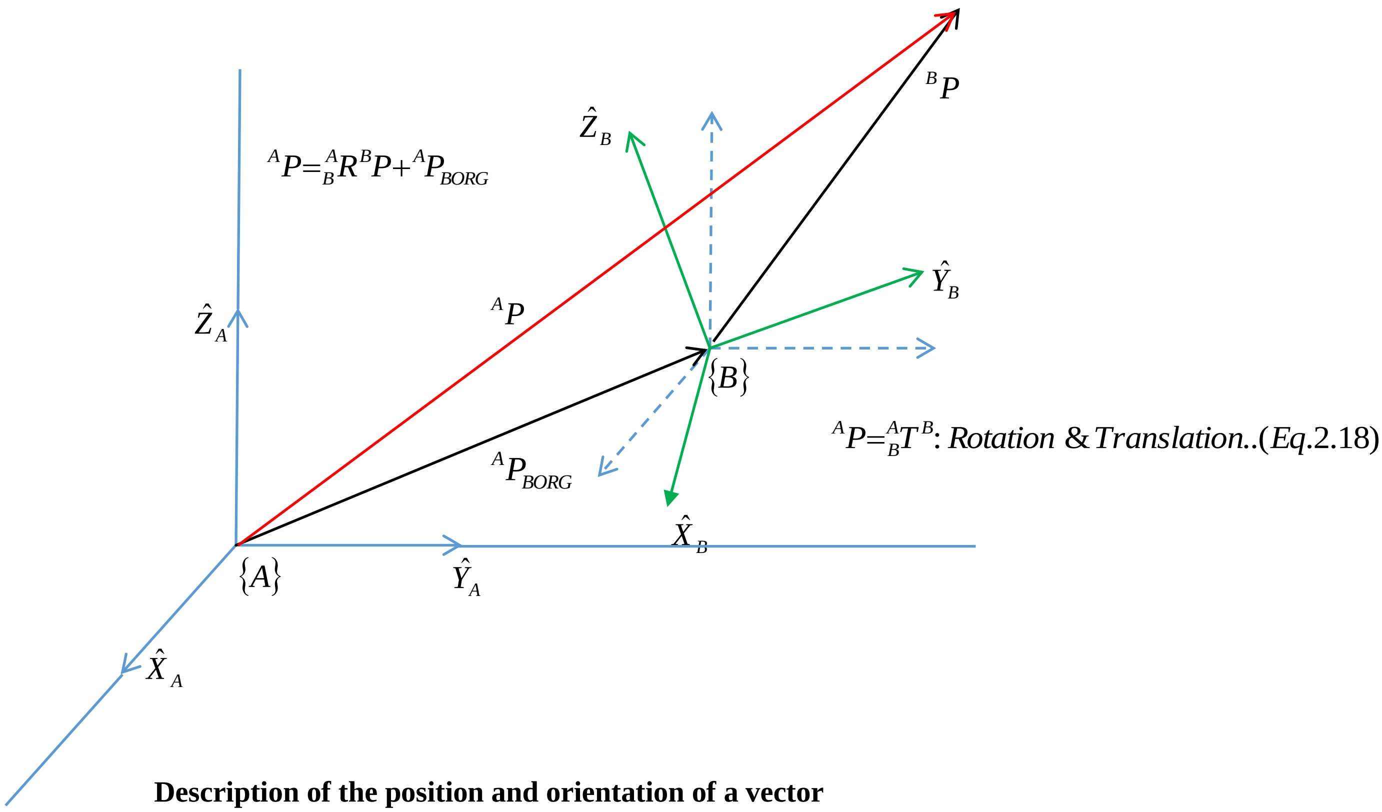
Homework No. 2 – Due 9/10/14

2.13 (Frame arrangement only. Do not calculate.), 2.21, 2.28, 2.31, 2.37

2.38 – Optional extra credit (A short proof that $R^T=R^{-1}$)

Homework No. 1 – Preassigned, due 9/3/14

Study Sections 2.1-2.4. Solve Problems: 2.1, 2.2, 2.3, 2.12, 2.17, 2.18. See PPT slides 16, 17. Use Matlab on 2.2 if possible. On 2.12, a rotation matrix also applies to velocity vectors.



Description of the position and orientation of a vector

Position vector P

$$\hat{P} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$$

Unit vector - Orientation in principal axes

$$[\hat{X},\hat{Y},\hat{Z}] = \frac{1}{\sqrt{p_x^2 + p_y^2 + p_z^2}}(p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$$

where $[P_x, P_y, P_z]$ is the directional cosine of P on X, Y, and Z axes.

$[\hat{X},\hat{Y},\hat{Z}]$ = the directional unit vector of P.

Orientation of the unit vector in frame

$$\{A\} = [\hat{X}_A, \hat{Y}_A, \hat{Z}_A]$$

Rotation matrix of {B} w.r.t. {A} is the projection of each axis of {B} onto each axis of {A}.
A dot product of two unit vectors is simply the direction cosine of the angle between the vectors.

$${}^{A}_{B}R = [{}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B] = \begin{bmatrix} \hat{X}_B \bullet \hat{X}_A & \hat{Y}_B \bullet \hat{X}_A & \hat{Z}_B \bullet \hat{X}_A \\ \hat{X}_B \bullet \hat{Y}_A & \hat{Y}_B \bullet \hat{Y}_A & \hat{Z}_B \bullet \hat{Y}_A \\ \hat{X}_B \bullet \hat{Z}_A & \hat{Y}_B \bullet \hat{Z}_A & \hat{Z}_B \bullet \hat{Z}_A \end{bmatrix} \quad (2.3)$$

- The rows are the rotated unit vectors in {B} expressed relative to the reference frame {A}.
- The columns are the unit vectors of the reference frame {A} expressed relative to the rotated {B}.
- As unit vectors, the magnitude of each row and column = 1 and therefore,

$$R^{-1} = R^T$$

$$RR^T = RR^{-1} = I_3 \quad (2.6)$$

$${}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} \quad (2.7)$$

Frame Description and Mapping

$$\{B\} = \{ {}^{A}_{B}R, {}^{A}P_{BORG} \} \quad (2.8)$$

Frame {B} may be constructed by (1) shifting its origin from that coinciding with reference frame {A} by vector ${}^{A}P_{BORG}$ and by (2) rotating its axes by ${}^{A}_{B}R$.

(1) Frame shift only: Vector P in {B} is expressed with respect to {A} through vector addition:

$${}^{A}P = {}^{B}P + {}^{A}P_{BORG} \quad (2.9)$$

The rotation matrix consists of three unit column vectors or three unit row vectors:

$${}^{A}_{B}R = [{}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B] = [{}^{B}\hat{X}_A{}^{T} \quad {}^{B}\hat{Y}_A{}^{T} \quad {}^{B}\hat{Z}_A{}^{T}]^T \quad (2.11)$$

(2) Frame rotation only: If ${}^{A}P_{BORG} = 0$, that is, the origins of frame {A} and {B} coincide, then

$${}^{A}P = {}^{A}_{B}R \; {}^{B}P \quad (2.13)$$

With both frame shift and rotation, vector P in {B} is given by

$${}^{A}P = {}^{A}_{B}R \, {}^{B}P + {}^{A}P_{BORG} \quad (2.17)$$

Homogenous Transform Matrix

A 4x4 composite transformation matrix T representing both frame shift and rotation:

$${}^{A}P' = {}^{A}_{B}T \; {}^{B}P'$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R_{3x3} & {}^{A}R_{BORG} \\ 0_{3x1} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} \tag{2.19}$$

Position Translation

If vector ${}^{A}P_1$ from the origin of {A} is translated by vector ${}^{A}Q$ also from the origin, define

$${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q \tag{2.24}$$

$${}^{A}P_2 = D_Q(q)\,{}^{A}P_1 \tag{2.25}$$

The translation matrix operator D_Q can be defined as

$$D_Q(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.26}$$

Two Step Transformation

If frame {c} is defined relative to {B} and {B} defined relative to {A}, then transformation matrix ${}^{A}_{B}T$ can be derived from Eq. (2.37) – (2.39).

$${}^{A}_{C}T = {}^{A}_{B}T\ {}^{B}_{C}T \tag{2.40}$$

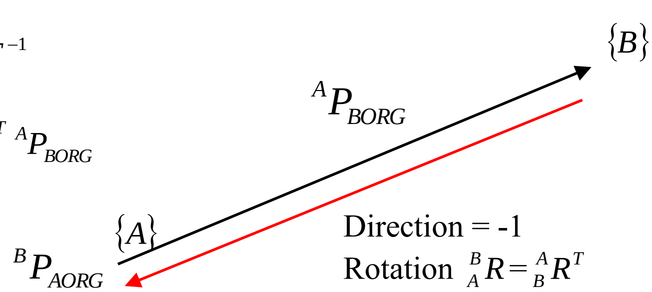
$${}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}R_{BORG} \\ 0_{3x1} & 1 \end{bmatrix} \tag{2.41}$$

The inverse of ${}^{A}_{B}T$ in (2.19):

$${}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ 0_{3x1} & 1 \end{bmatrix} \tag{2.46}$$

and $\quad {}^{B}_{A}T = {}^{A}_{B}T^{-1}$

Reasoning for the position vector $-{}^{A}_{B}R^{T}\,{}^{A}P_{BORG}$



Denavit-Hartenberg Parameters (from Chapter 3)

α_{i-1}	a_{i-1}	d_i	θ_i

α_{i-1} =Link twist (0 or ±90°) *a_{i-1}=Link Length* *d_i=X axis offset* *θ_i=Z rotation about X*

Link Transformation

Applying the Denavit-Hartenberg parameters in sequential transformation,

$$ {}^{i-1}_{\;\;i}T = \underbrace{Rx(\alpha_{i-1})\cdot Dx(a_{i-1})}_{\text{Rotate, then translate in \{i-1\}}}\cdot \underbrace{Rz(\theta_i)\cdot Dz(d_i)}_{\text{Rotate, then translate in \{i\}}} $$

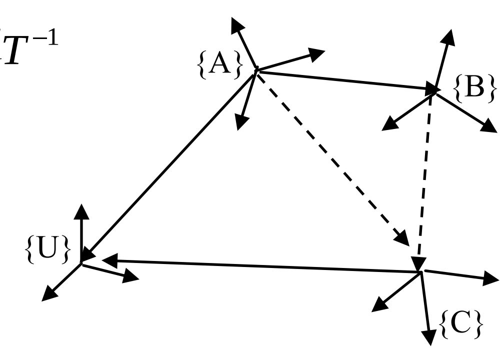
$$ = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}} $$

$$ {}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & -s\alpha_{i-1} d_i \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.6) $$

Transform Equations

Successive transformation of a position vector may be performed in a forward and reverse direction.

$$ {}^{B}_{C}T = {}^{B}_{A}T\,{}^{U}_{A}T^{-1}\,{}^{C}_{U}T^{-1} $$



To find ${}^{A}_{C}T$: $\quad {}^{U}_{A}T^{-1}\,{}^{C}_{U}T^{-1} = {}^{A}_{U}T\,{}^{U}_{C}T = {}^{B}_{A}T^{-1}\,{}^{B}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T = {}^{A}_{C}T$

Using $\quad {}^{Q}_{p}T = {}^{P}_{Q}T^{-1} = {}^{P}_{Q}T^{T} \qquad (RS)^T = S^T R^T$

Other Ways of Rotation Description – Due to the Properties of Orthonormal Rotation Axes

(A) Skew Symmetric Matrix

Rotational matrices are ***proper*** orthonormal matrices as their determinants are 1. As such, a skew matrix in which $S + S^T = 0$, exists for each rotational matrix.

$$R = (I_3 - S)^{-1} (I_3 + S) \tag{2.56}$$

S has three parameters (σ_x, σ_y, σ_z). Thus, R may be represented by $\bar{\sigma} = [\sigma_x, \sigma_y, \sigma_z]^T$. For application, in joint angular velocity analysis, it can be shown that for a given R, a product of R and its derivative forma a skew symmetric matrix,

$$S = \dot{R}R^T \tag{5.17}$$

and has the form:

$$\mathbf{S} = \begin{bmatrix} 0 & -\sigma_Z & \sigma_y \\ \sigma_Z & 0 & -\sigma_x \\ -\sigma_y & \sigma_x & 0 \end{bmatrix} \tag{2.57}$$

Given any position vector P, S has following property:

$$SP = \bar{\sigma} \times P \tag{5.27}$$

where $\bar{\sigma} \times P$ is a vector cross product and represents the angular velocity of a rotating axis.

The application can be extended to the case of rotation about a general axis $\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}$.

(B) X-Y-Z Fixed Angles

Frame {A} and {B} are coincident. Rotate {B} about $\hat{X}_A$ by γ, the about Y $\hat{Y}_A$ by β, and about $\hat{Z}_A$ by α.

$${}^A_B R_{xyz}(\gamma, \beta, \alpha) = R_Z(\alpha) R_Y(\beta) R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \tag{2.63}$$

$${}^A_B R_{xyz}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \tag{2.64}$$

$${}^A_B R_{xyz}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The solutions can be found using trigonometric identities, sum of squares of sine and cosine angles, and the tangent angles on the terms in (2.64):

$$\beta = A\tan 2(-r_{31}, \sqrt{r_{11}^{2} + r_{12}^{2}}$$
$$\alpha = A\tan 2(r_{21}/c\beta, r_{11}/c\beta) \quad (2.66)$$
$$\gamma = A\tan 2(r_{32}/c\beta, r_{33}/c\beta)$$

(C1) Z-Y-X Euler Angles

Frames {A} and {B} are coincident. Rotate {B} about $\hat{Z}_B$ by α, then about $\hat{Y}_B$ by β, and then $\hat{X}_B$ by γ.

$${}^{A}_{B}R_{z'y'x'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \quad (2.70)$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \quad (2.71)$$

The result is the same as (2.64). The results from rotations about three fixed axes is the same as the results from rotations about the moving frames taken in reverse order.

(C2) Z-Y-Z Euler Angles

Applicable when three axes intersect as in the case of the wrist joints where yaw, pitch, and roll axescome together (Joint 4 and 5 are orthogonal and Joint 5 and 6 are coaxial.)

Frames {A} and {B} are coincident. Rotate {B} about $\hat{Z}_B$ by α, then about $\hat{Y}_B$ by β, and then $\hat{Z}_b$ by γ.

$${}^{A}_{B}R_{z'y'z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (2/72)$$

$$\beta = A\tan 2(\sqrt{r_{31}^{2} + r_{32}^{2}}, r_{33})$$
$$\alpha = A\tan 2(r_{23}/s\beta, r_{13}/s\beta) \quad (2.74)$$
$$\gamma = A\tan 2(r_{32}/s\beta, -r_{31}/s\beta)$$

Other Angle-Set Conventions

24 total possible. Found in Appendix B

(D) Rotation about Equivalent Angle-Axis (Euler Angle Axis)

Frames {A} and {B} are coincident. Rotate {B} about vector $\hat{K}_A$ by θ following the right hand rule.

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \qquad R_Y(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \qquad R_Z(\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.77\text{-}79)$$

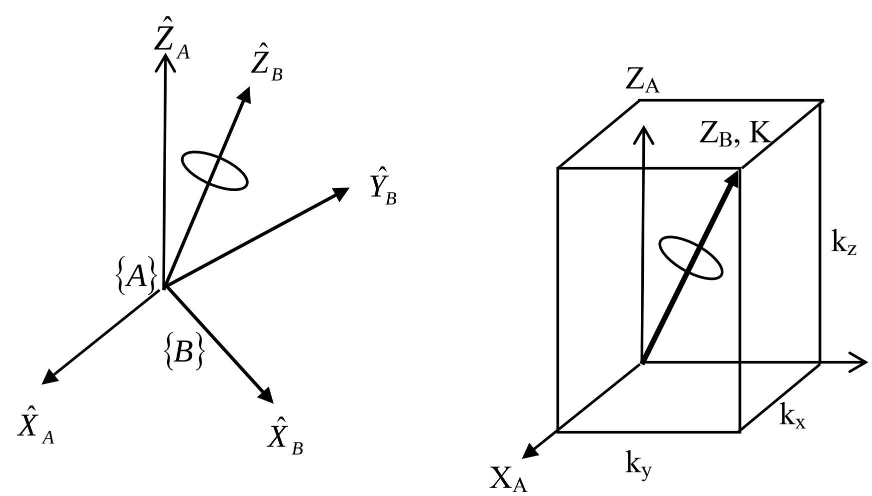
$$R_K(\theta) = \begin{bmatrix} k_x k_x \upsilon\theta + c\theta & k_x k_y \upsilon\theta - k_z s\theta & k_x k_z \upsilon\theta + k_y s\theta \\ k_x k_y \upsilon\theta + k_z s\theta & k_y k_y \upsilon\theta + c\theta & k_y k_z \upsilon\theta - k_x s\theta \\ k_x k_z \upsilon\theta - k_y s\theta & k_y k_z \upsilon\theta + k_x s\theta & k_z k_z \upsilon\theta + c\theta \end{bmatrix} \qquad (2.80)$$

$${}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad (2.81)$$

$$= \text{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2} \right)$$

$$\hat{K} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \qquad (2.82)$$

Equivalent Angle Rotation Axis (Eq. 2.80)



Eq. 2.80. Solve:

$${}^{A}_{B}R \cdot rot({}^{A}\hat{Z}, \theta) \cdot {}^{A}_{B}R^{-1} = {}^{A}_{B}R \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^{B}_{A}R, \quad \text{where } {}^{A}_{B}R(\alpha) = \begin{bmatrix} a_{11} & a_{12} & k_x \\ a_{21} & a_{22} & k_Y \\ a_{31} & a_{32} & k_Z \end{bmatrix}$$

where $[K_x, K_y, K_z]^T$ is a column vector representing the project of K onto the principal axes of {A}.

Steps

1) Make frame {B} to be rotated coincide with frame {A}.
2) Tilt {B} away from {A} so that Z_B will coincide with K, the equivalent axis. → ${}^{A}_{B}R$
3) Rotate {B} about Z_B by θ.
4) Make the rotation appear that it was done with reference to a fixed axis of {A}

$$\rightarrow R = {}^{A}_{B}R \cdot rot({}^{B}\hat{Z},\theta) \cdot {}^{A}_{B}R^{-1}$$ ← See Problem 2.19 for similarity matrix.

5) Make substitutions using

$$k_x^2 + k_y^2 + k_z^2 = 1 \quad a_{11}^2 + b_{21}^2 + c_{31}^2 = 1 \quad a_{21}^2 + b_{22}^2 + c_{32}^2 = 1$$
$$(a_{11}, b_{21}, c_{31}) \bullet (a_{21}, b_{22}, c_{32}) = 0 \quad (a_{11}, b_{21}, c_{31}) \times (a_{21}, b_{22}, c_{32}) = (k_x, k_y, k_z)$$

(E) Euler Parameters for Equivalent Axis

Associated with equivalent axis and angle of rotation $\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}$ and *θ*.

Euler parameters (Unit Quarternion) = [$\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4$], where

$$\varepsilon_1 = k_x \sin\frac{\theta}{2} \qquad \varepsilon_2 = k_y \sin\frac{\theta}{2} \qquad \varepsilon_3 = k_z \sin\frac{\theta}{2} \qquad \varepsilon_4 = \cos\frac{\theta}{2} \tag{2.89}$$

Then $\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$ (2.90)

$$R_\varepsilon = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix} \tag{2.91}$$

Problem: Verify that (2.91) and (2.80) yields the same rotation.

Solution: Replace the Euler parameters in (2.91) with the terms in (2.89) to arrive at (2.80).
Use the relations $Cos\theta = 2Cos_{\theta/2}^2 - 1 = 1 - 2\sin_{\theta/2}^2$

For a given rotation matrix, the direction vector $\hat{K}$ and θ are found by extracting the Euler parameters:

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4} \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4} \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4} \tag{2.92}$$
$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$