Worksheet 10: Dimension and rank

Example 0.66. Let $\mathbf{v}_1 = [1,1,0]^T, \mathbf{v}_2 = [1,0,1]^T$. Then $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^3$. It follows that the dimension of H is 2, i.e., $\dim(H) = 2$.

Example 0.67. For the following matrix, $\dim(\text{Col}(\mathbf{A})) = 3$ (pivot columns), and $\dim(\text{Nul}(\mathbf{A})) = 2$ (free variables).

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 0.68. For the matrix $\mathbf{A}$ defined above, its rank is 3. Thus, the maximal number of linearly independent columns is also 3.

Example 0.69. Consider the above matrix again: Because its rank is 3, we must have

$$\dim(\text{Nul}(\mathbf{A})) = n - \text{rank}(\mathbf{A}) = 5 - 3 = 2.$$

You may want to verify this by finding a basis for the null space:

Example 0.70. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. The column space is the span of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, while the row space is the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$.

Example 0.71. The following two matrices have the same row space, but not the same column space:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \quad \longrightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The reason is that linear combinations of rows of $\mathbf{B}$, which are linear combinations of rows of $\mathbf{A}$, are always linear combinations of rows of $\mathbf{A}$ (and vice versa).

Example 0.72. Find a basis for the row space of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 0.73 (p233)**.** Find a basis for each of the row/column/null spaces of the following matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \longrightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$