San José State University

Math 161a: Applied Probability & Statistics

Lecture 2: Counting

Prof. Guangliang Chen

Section 2.3 Counting Techniques

Introduction

Knowing how to count is very important in the study of probability, as it is often needed to count the objects in a sample space, or those in a subset (i.e. event).

For example, in the setting of **a finite sample space with equally likely outcomes**, the formula for computing the probability of any event $E \subset S$ involves two counting tasks:

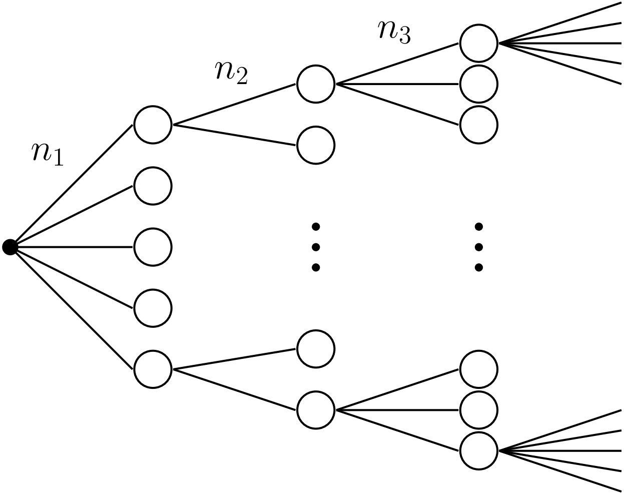
$$P(E) = \frac{|E|}{|S|}.$$

Fundamental Counting Principle

Theorem 0.1. Suppose an experiment can be performed in a sequence of k steps, such that

- the first step can be done in n_1 ways, and
- for each result of step 1, step 2 can always be done in n_2 ways, and
- step 3 can always be done in n_3 ways for each combination of results of steps 1 and 2, so on and so forth.

Then the entire experiment has a total of $n_1 n_2 \cdots n_k$ possible outcomes.


n_1
n_2
n_3

Example 0.1. A local restaurant provides 5 kinds of bread, 4 kinds of cheese, 4 kinds of meats, and 6 kinds of sauces. In how many ways can you order a sandwich?

Example 0.2. How many different CA driver license numbers are there (1 capital letter followed by 7 digits)? How many driver license numbers have all repeated digit? All distinct digits?

Solution:

$$26 \cdot \underbrace{10 \cdot 10 \cdots 10}_{\text{7 times}} = 260,000,000$$

$$26 \cdot 10 \cdot \underbrace{1 \cdots 1}_{\text{6 copies}} = 260$$

$$26 \cdot \underbrace{10 \cdot 9 \cdots 4}_{\text{7 numbers}} = 15,724,800$$

Example 0.3. How many 3-digit numbers are divisible by 5?

Permutations

Briefly, permutations are ordered lists consisting of distinct objects, e.g.,

$$\{0, 1, 2, \ldots, 9\} \quad \longrightarrow \quad 5810,\ 105,\ 043987,\ 71,\ 3028971,\ 16345, \ldots$$

Def 0.1. A **permutation of size** r chosen from a set of n objects is an ordered list of r distinct objects from the set (without replacement).

_____	_____	_____	_____	_____
position 1	position 2			position r

Example 0.4. List all permutations of size $r = 3$ chosen from the set $S = \{a, b, c, d\}$. How many are there? What if $r = 4$?

Theorem 0.2. The number of permutations of size r that can be formed from a total of n objects is

$$P(n,r) = \underbrace{n(n-1)\cdots(n-r+1)}_{r \text{ integers}} = \frac{n!}{(n-r)!}.$$

In particular,

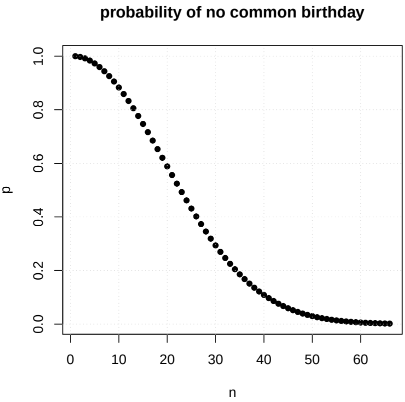
$$P(n,n) = n! \quad \longleftarrow \text{\#full permutations of size } n$$

Example 0.5. In how many different ways can 5 people be arranged in a row? Along a circle?

Example 0.6. Suppose you have 10 textbooks, in which 5 are about math, 3 about computer science and 2 about English. In how many different ways can you arrange them in a line to put on your bookshelf? What if you want to have the books of the same subject all together?

Example 0.7 (Birthday problem). Assume that people's birthdays are equally likely to occur among the 365 days of the year and ignore leap years. Find the probability p that no two people in a class of 35 have a common birthday, i.e., all students have different birthdays.

(Answer: $.1856$.)



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Combinations

Briefly, combinations are unordered collections of distinct objects, e.g.,

$$\{0,1,2,\ldots,9\} \quad \longrightarrow \quad \{0,1,5,8\},\ \{0,3,4,9\},\ \{1,2,7,9\},\ldots$$

Def 0.2. A **combination of size** r chosen from a set of n objects is an unordered selection of r distinct objects from the set (without replacement).

Example 0.8. List all combinations of size 3 chosen from the set $S = \{a,b,c,d\}$.

Theorem 0.3. The number of combinations of size r that can be formed from a total of n objects is

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)! \cdot r!}.$$

Remark. To compute $\binom{n}{r}$ by hand, use the following equivalent formula (and make cancellations as much as possible):

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r} \qquad \longleftarrow \frac{\text{"largest } r\text{"}}{\text{"smallest } r\text{"}}$$

In particular,

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{r} = \binom{n}{n-r} \text{ for any } 0 \le r \le n$$

$$\binom{n}{n} = 1$$

Example 0.9. Consider the problem of choosing 4 members from a group of 10 to work on a special project.

(a) Suppose two people A and B really like each other, so they must be simultaneously chosen or skipped. How many distinct four-person teams can be chosen?

(b) Suppose two people A and B really hate each other, so they cannot be both selected for the project. How many distinct four-person teams can be chosen?

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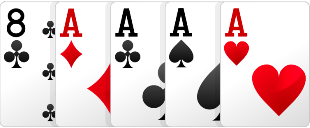
Example 0.10. An urn has 5 red balls and 7 blue balls. Suppose you randomly select 5 balls from the urn. What is the probability that your hand has exactly 3 red balls?

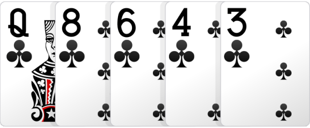
A ordinary deck of 52 cards is divided into 4 suits (heart, diamond, spade and club) and 13 ranks (2, 3, ..., 10, J, Q, K, A)

Example 0.11. Suppose your randomly draw 5 cards from a deck of 52. What is the probability that you have a

(a) four of a kind (4 cards of the same rank, and one side card)

(b) flush (5 cards of the same suit)





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Summary

We covered the following material during this lecture:

- **Fundamental Counting Principle**

- **Permutations** (ordered lists of distinct objects): Given a set of n objects, the total number of permutations of size r that can be formed from the set is

$$P(n,r) = \underbrace{n(n-1)\cdots(n-r+1)}_{r \text{ integers}} = \frac{n!}{(n-r)!}$$

- **Combinations** (unordered lists of distinct objects): Given a set of n objects, the total number of combinations of size r that can be formed from the set is

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n!}{(n-r)!\cdot r!}$$